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Common fixed point results from quasi-metric spaces to G-metric spaces

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Abstract In this paper, we provide some common fixed point results involving implicit contractions on quasi-metric spaces, and based on the recent nice paper of Jleli and Samet (2012), we show that some common fixed point theorems involving implicit contractions on *G*-metric spaces can be deduced immediately from our common fixed point theorems on quasi-metric spaces. The notion of well-posedness of the common fixed point problem is also studied.

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1. Introduction and preliminaries

It is well known that passing from metric spaces to quasi-metric spaces carries with it immediate consequences to the general theory. The definition of a quasi-metric is given as follows:

Definition 1.1. Let X be a non-empty and let $d: X \times X \to [0, \infty)$ be a function which satisfies:

$(d1)\, d(x, y) = 0$ if and only if $x = y$,

$(d2)\, d(x, y) \leqslant d(x, z) + d(z, y)$. Then d is called a quasi-metric and the pair (X, d) is called a quasi-metric space.

Remark 1.1. Any metric space is a quasi-metric space, but the converse is not true in general.

Now, we give convergence and completeness on quasi-metric spaces.

Definition 1.2. Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0. \tag{1}$$

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Definition 1.3. Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geqslant m > N$.

Definition 1.4. Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geqslant n > N$.

Definition 1.5. Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Remark 1.2. A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 1.6. Let (X, d) be a quasi-metric space. We say that

(1) (X, d) is left-complete if and only if each left-Cauchy sequence in X is convergent.

(2) (X, d) is right-complete if and only if each right-Cauchy sequence in X is convergent.

(3) (X, d) is complete if and only if each Cauchy sequence in X is convergent.

The following definitions and results are also needed in the sequel.

Definition 1.7. Let f and g be self maps of a non-empty set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Definition 1.8. Let f and g be self maps of a non-empty set X. If f and g commute at their coincidence points, then they called weakly compatible mappings.

Lemma 1.1. [1] *Let f and g be weakly compatible self mappings of non-empty set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

On the other hand, the study of fixed point for mappings satisfying an implicit relation is initiated and studied by Popa [2,3]. It leads to interesting known fixed points results. Following Popa's approach, many authors proved some fixed point, common fixed point and coincidence point results in various ambient spaces, see [4–7].

In the literature, there are several types of implicit contraction mappings where many nice consequences of fixed point theorems could be derived. For instance, Popa and Patriciu [8] introduced the following

Definition 1.9. [8] Let Γ_0 be the set of all continuous functions $F(t_1, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that

$(A1)$: F is non-increasing in variable t_5,

$(A2)$: There exists a certain function h_1 such that for all $u, v \geqslant 0, F(u, v, v, u, u + v, 0) \leqslant 0$ implies $u \leqslant h_1(v)$,

$(A3)$: There exists a certain function h_2 such that for all $t, s > 0, F(t, t, 0, 0, t, s) \leqslant 0$ implies $t \leqslant h_2(s)$.

We denote Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:

(ψ_1) ψ is non-decreasing,

(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where ψ^n is the *nth* iterate of ψ.

Remark 1.3. It is easy to see that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

We introduce the following Definition.

Definition 1.10. Let Γ be the set of all continuous functions $F(t_1, \ldots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that

$(F1)$: F is non-increasing in variable t_5,

$(F2)$: There exists $h_1 \in \Psi$ such that for all $u, v \geqslant 0, F(u, v, v, u, u + v, 0) \leq 0$ implies $u \leqslant h_1(v)$,

$(F3)$: There exists $h_2 \in \Psi$ such that for all $t, s > 0, F(t, t, 0, 0, t, s) \leqslant 0$ implies $t \leqslant h_2(s)$.

Note that in Definition 1.10, we did not take the same hypotheses on h_1 and h_2 as in Definition 1.9, that is, some ones are dropped. As in [8], we give the following examples.

Example 1.1. $F(t_1, \ldots, t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \geqslant 0, a + b + c + 2d + e < 1$.

$(F1)$: Obvious.

$(F2)$: Let $u, v \geqslant 0$ and $F(u, v, v, u + v, 0) = u - av - bv - cu - d(u + v) \leqslant 0$ which implies $u \leqslant \frac{a+b+d}{1-c-d} v$ and $(F2)$ is satisfied for $h_1(t) = \frac{a+b+d}{1-(c+d)} t$.

$(F3)$: Let $t, s > 0$ and $F(t, t, 0, 0, t, s) = t - at - dt - es \leqslant 0$ which implies $t \leqslant \frac{e}{1-(a+d)} s$ and $(F3)$ is satisfied for $h_2(s) = \frac{e}{1-(a+d)} s$.

Example 1.2. $F(t_1, \ldots, t_6) = t_1 - k \max\{t_2, \ldots, t_6\}$, where $k \in \left[0, \frac{1}{2}\right)$.

$(F1)$: Obvious.

$(F2)$: Let $u, v \geqslant 0$ and $F(u, v, v, u, u + v, 0) = u - k \max\{u, v, u + v\} \leq 0$. Thus, $u \leqslant \frac{k}{1-k} v$ and $(F2)$ is satisfied for $h_1(t) = \frac{k}{1-k} t$.

$(F3)$: Let $t, s > 0$ and $F(t, t, 0, 0, t, s) = t - k \max\{t, s\} \leq 0$. If $t > s$, then $t(1 - k) \leqslant 0$, a contradiction. Hence $t \leqslant s$ which implies $t \leqslant ks$ and $(F3)$ is satisfied for $h_2(s) = ks$.

Some other examples could be derived from [8].

In this paper, we provide some common fixed point results involving implicit contractions on quasi-metric spaces. We also prove the posedness of the common fixed point problem. Finally, we show that some existing fixed point results on G-metric spaces are immediate consequences of our main presented theorems on quasi-metric spaces.

2. Fixed point theorems

In this section we shall state and prove our main results. We first prove the uniqueness of a common fixed point of certain operators if it exists.

Lemma 2.1. *Let (X,d) be a quasi-metric space and $f,g:(X,d)\to(X,d)$ two functions such that*

$$F(d(fx,fy),d(gx,gy),d(gx,fx),d(gy,fy),d(gx,fy),d(gy,fx)) \leqslant 0,\ \forall x,y\in X \tag{2}$$

and F satisfying property $(F3)$. Then, f and g have at most one point of coincidence.

Proof. We assume that f and g have two points of coincidence u and v $(u\neq v)$. In this case, there exist $p,q\in X$ such that $u=fp=gp$ and $v=fq=gq$. Then by using (2) we get

$$F(d(fp,fq),d(gp,gq),d(gp,fp),d(gq,fq),d(gp,fq),d(gq,fp)) \leqslant 0,$$

that is

$$F(d(gp,gq),d(gp,gq),0,0,d(gp,gq),d(gq,gp)) \leqslant 0.$$

Since F satisfies property $(F3)$, so

$$d(gp,gq) \leqslant h_2(d(gq,gp)). \tag{3}$$

Analogously, we obtain

$$d(gq,gp) \leqslant h_2(d(gp,gq)). \tag{4}$$

Combining (3) and (4), we get using the fact that h_2 is non-decreasing and $h_2(t)<t$ for $t>0$

$$0<d(gp,gq)\leqslant h_2(d(gq,gp))\leqslant h_2^2(d(gp,gq)<d(gp,gq)). \tag{5}$$

It is a contradiction. Hence $gp=gq$. Therefore $u=fp=gp=gq=fq=v$. □

In what follows that we prove the existence of a common fixed point of two self-mappings under certain implicit relations.

Theorem 2.1. *Let (X,d) be a quasi-metric space and $f,g:(X,d)\to(X,d)$ satisfying inequalities*

$$F(d(fx,fy),d(gx,gy),d(gx,fx),d(gy,fy),d(gx,fy),d(gy,fx))\leqslant 0, \tag{6}$$

for all $x,y\in X$, where $F\in\Gamma$. If $f(X)\subseteq g(X)$ and $g(X)$ is a complete quasi metric subspace of (X,d), then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X and by using $f(X)\subseteq g(X)$ we can choose $x_1\in X$ such that $fx_0=gx_1$. If we keep this up, we obtain x_{n+1} such that $fx_n=gx_{n+1}$. Then by (6) we have

$$F(d(fx_{n-1},fx_n),d(gx_{n-1},gx_n),d(gx_{n-1},fx_{n-1}),d(gx_n,fx_n),\\ d(gx_{n-1},fx_n),d(gx_n,fx_{n-1}))\leqslant 0,$$

that is,

$$F(d(gx_n,gx_{n+1}),d(gx_{n-1},gx_n),d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),\\ d(gx_{n-1},gx_{n+1}),0)\leqslant 0.$$

By $(F1)$ and $(d2)$, we have

$$F(d(gx_n,gx_{n+1}),d(gx_{n-1},gx_n),d(gx_{n-1},gx_n),d(gx_n,gx_{n+1}),\\ d(gx_{n-1},gx_n)+d(gx_n,gx_{n+1}),0)\leqslant 0. \tag{7}$$

By $(F2)$, we obtain

$$d(gx_n,gx_{n+1})\leqslant h_1(d(gx_{n-1},gx_n)). \tag{8}$$

If we go on like this, we get

$$d(gx_n,gx_{n+1})\leqslant h_1^n(d(gx_0,gx_1)). \tag{9}$$

Thus, by using $(d2)$, for $m>n$

$$\begin{aligned} d(gx_n,gx_m) &\leqslant d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})+\cdots+d(gx_{m-1},gx_m)\\ &\leqslant (h_1^n+h_1^{n+1}+\cdots+h_1^{m-1})(d(gx_0,gx_1))\\ &\leqslant \tfrac{h_1^n}{1-h_1}(d(gx_0,gx_1)), \end{aligned} \tag{10}$$

which implies that $d(gx_n,gx_m)\to 0$ as, $n,m\to\infty$. It follows that $\{gx_n\}$ is a right-Cauchy sequence.

Similarly, by (6) we have

$$F(d(fx_n,fx_{n-1}),d(gx_n,gx_{n-1}),d(fx_{n-1},gx_{n-1}),d(fx_n,gx_n),\\ d(fx_n,gx_{n-1}),d(fx_{n-1},gx_n))\leqslant 0,$$

that is,

$$F(d(gx_{n+1},gx_n),d(gx_n,gx_{n-1}),d(gx_n,gx_{n-1}),d(gx_{n+1},gx_n),\\ d(gx_{n+1},gx_{n-1}),0)\leqslant 0.$$

Using $(F1)$ and $(d2)$

$$F(d(gx_{n+1},gx_n),d(gx_n,gx_{n-1}),d(gx_n,gx_{n-1}),d(gx_{n+1},gx_n),\\ d(gx_{n+1},gx_n)+d(gx_n,gx_{n-1}),0)\leqslant 0. \tag{11}$$

By $(F2)$ we obtain

$$d(gx_{n+1},gx_n)\leqslant h_1(d(gx_n,gx_{n-1})). \tag{12}$$

If we go on like this, we get

$$d(gx_{n+1},gx_n)\leqslant h_1^n(d(gx_1,gx_0)). \tag{13}$$

Thus, by using $(d2)$, for $n>m$

$$\begin{aligned} d(gx_n,gx_m) &\leqslant d(gx_n,gx_{n-1})+d(gx_{n-1},gx_{n-2})+\cdots+d(gx_{m+1},gx_m)\\ &\leqslant \left(h_1^{n-1}+h_1^{n-2}+\cdots+h_1^m\right)(d(gx_1,gx_0))\\ &\leqslant \tfrac{h_1^m}{1-h_1}(d(gx_1,gx_0)), \end{aligned} \tag{14}$$

which implies that $d(gx_n,gx_m)\to 0$ as, $n,m\to\infty$. It follows that $\{gx_n\}$ is a left-Cauchy sequence.

Thus, $\{gx_n\}$ is a Cauchy sequence. Since $g(X)$ is quasi-complete, there exists a point $q=gp$ in $g(X)$ such that $gx_n\to q=gp$ as $n\to\infty$. We shall prove that $fp=gp$.

By (6), we have successively

$$F(d(fx_{n-1},fp),d(gx_{n-1},gp),d(gx_{n-1},fx_{n-1}),d(gp,fp),\\ d(gx_{n-1},fp),d(gp,fx_{n-1}))\leqslant 0,$$

that is,

$$F(d(gx_n,fp),d(gx_{n-1},gp),d(gx_{n-1},gx_n),d(gp,fp),d(gx_{n-1},fp),\\ d(gp,gx_n))\leqslant 0.$$

Letting n tend to infinity, we have

$$F(d(gp,fp),0,0,d(gp,fp),d(gp,fp),0) \leqslant 0.$$

By $(F2)$, it follows that $d(gp,fp)=0$ which implies $gp=fp$. Hence $w=fp=gp$ is a point of coincidence of f and g. By using Lemma 2.1, w is the unique point of coincidence. Moreover, since f and g are weakly compatible, so by Lemma 1.1, w is the unique common fixed point of f and g. □

In the sequel, we present the following corollaries as consequences of Theorem 2.1.

Corollary 2.1. *Let (X,d) be a complete quasi-metric space. Suppose that*

$$F(d(fx,fy),d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)) \leqslant 0 \quad (15)$$

holds for all $x,y \in X$ where $F \in \Gamma$. Then f has a unique fixed point.

Proof. If we choose g the identity function, then by Theorem 2.1, it is easy that f has a unique fixed point. □

The following corollary is a Ćirić contraction type [9].

Corollary 2.2. *Let (X,d) be a quasi-metric space and $f,g:(X,d)\to(X,d)$ satisfying*

$$d(fx,fy) \leqslant k\max\{d(gx,gy),d(gx,fx),d(gy,fy),d(gx,fy),\\ d(gy,fx)\}, \quad (16)$$

for all $x,y \in X$, where $k \in \left[0,\frac{1}{2}\right)$. If $f(X)\subseteq g(X)$ and $g(X)$ is a complete quasi metric subspace of (X,d), then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. It suffices to take F as given in Example 1.2, that is, $F(t_1,\ldots,t_6)=t_1-k\max\{t_2,\ldots,t_6\}$, where $k\in[0,\frac{1}{2})$. Then, we apply Theorem 2.1. □

Remark 2.1. Theorem 2.1 (resp. Corollary 2.1) is an extension of Theorem 1 (Corollary 1) of Berinde and Vetro [10] to quasi-metric spaces.

3. Well posedness problem of fixed point for two mappings in quasi metric spaces

The notion of well-posedness of a fixed point has evoked much interest to several mathematicians, as example see [11–13]. We start to characterize the concept of the well-posedness in the context of quasi-metric spaces in the following way.

Definition 3.1. Let (X,d) be a quasi-metric space and $f:(X,d)\to(X,d)$ be a given mapping. The fixed point problem f is said to be well posed if

(1) f has a unique fixed point $x_0 \in X$,
(2) for any sequence $\{x_n\}\subseteq X$ with $\lim_{n\to\infty}d(x_n,fx_n)=\lim_{n\to\infty}d(fx_n,x_n)=0$, then we have $\lim_{n\to\infty}d(x_n,x_0)=\lim_{n\to\infty}d(x_0,x_n)=0$.

We also need the following definition.

Definition 3.2. A function $F:\mathbb{R}_+^6\to\mathbb{R}$ has property (F_p) if for $u,v,w\geqslant 0$ and $F(u,v,0,w,u,v)\leq 0$, there exists $p\in(0,1)$ such that $u\leqslant p\max\{v,w\}$.

We introduce the notion well-posedness of a common fixed point problem on quasi-metric spaces as follows.

Definition 3.3. Let (X,d) be a quasi-metric space and $f,g:(X,d)\to(X,d)$. The common fixed problem of f and g is said to be well posed if

(1) f and g have a unique common fixed point,
(2) for any sequence $\{x_n\}\subseteq X$ with

$$\begin{aligned}&\lim_{n\to\infty}d(x_n,fx_n)=\lim_{n\to\infty}d(fx_n,x_n)=0 \text{ and}\\ &\lim_{n\to\infty}d(x_n,gx_n)=\lim_{n\to\infty}d(gx_n,x_n)=0,\end{aligned} \quad (17)$$

then $\lim_{n\to\infty}d(x,x_n)=\lim_{n\to\infty}d(x_n,x)=0$.

Our second main result is

Theorem 3.1. *Let (X,d) be a quasi-metric space. Assume that $f,g:(X,d)\to(X,d)$ satisfy hypotheses of Theorem 2.1 and F has property (F_p). Then, the common fixed point problem of f and g is well posed.*

Proof. By Theorem 2.1, f and g have a unique common fixed point x. Let $\{x_n\}$ be a sequence in (X,d) such that

$$\begin{aligned}&\lim_{n\to\infty}d(x_n,fx_n)=\lim_{n\to\infty}d(fx_n,x_n)=0 \text{ and}\\ &\lim_{n\to\infty}d(x_n,gx_n)=\lim_{n\to\infty}d(gx_n,x_n)=0.\end{aligned} \quad (18)$$

By (6), we have

$$F(d(fx,fx_n),d(gx,gx_n),d(gx,fx),d(gx_n,fx_n),d(gx,fx_n),\\ d(fx,gx_n)) \leqslant 0,$$

so

$$F(d(x,fx_n),d(x,gx_n),0,d(gx_n,fx_n),d(x,fx_n),d(x,gx_n)) \leqslant 0.$$

Using (F_p) property, we have

$$\begin{aligned}d(x,fx_n) &\leqslant p\max\{d(x,gx_n),d(gx_n,fx_n)\}\\ &\leqslant p(d(x,gx_n)+d(gx_n,fx_n)).\end{aligned} \quad (20)$$

Then by $(d2)$, we get

$$\begin{aligned}d(x,x_n) &\leqslant d(x,fx_n)+d(fx_n,x_n)\\ &\leqslant p(d(x,gx_n)+d(gx_n,fx_n))+d(fx_n,x_n)\\ &\leqslant p(d(x,x_n)+d(x_n,gx_n)+d(gx_n,x_n)+d(x_n,fx_n))\\ &\quad +d(fx_n,x_n).\end{aligned}$$

Thus

$$d(x,x_n) \leqslant \frac{p}{1-p}(d(x_n,gx_n)+d(gx_n,x_n)+d(x_n,fx_n))\\ +\frac{1}{1-p}d(fx_n,x_n). \quad (21)$$

Taking limit as $n\to\infty$ in (21) we obtain $\lim_{n\to\infty}d(x,x_n)=0$.

Similarly, by (6)

$$F(d(fx_n,fx),d(gx_n,gx),d(fx,gx),d(fx_n,gx_n),\\ d(fx_n,gx),d(gx_n,fx))\leqslant 0, \quad (22)$$

so

$$F(d(fx_n,x),d(gx_n,x),0,d(fx_n,gx_n),d(fx_n,x),d(gx_n,x))\leqslant 0.$$

Using (F_p) property, we have

$$\begin{aligned} d(fx_n,x) &\leqslant p\max\{d(gx_n,x),d(fx_n,gx_n)\}\\ &\leqslant p(d(gx_n,x)+d(fx_n,gx_n)). \end{aligned} \quad (23)$$

Then by $(d2)$, we get

$$\begin{aligned} d(x_n,x) &\leqslant d(x_n,fx_n)+d(fx_n,x)\\ &\leqslant d(x_n,fx_n)+p(d(gx_n,x)+d(fx_n,gx_n))\\ &\leqslant d(x_n,fx_n)+p(d(gx_n,x_n)+d(x_n,x)+d(fx_n,x_n)\\ &\quad +d(x_n,gx_n)). \end{aligned} \quad (24)$$

Thus

$$d(x_n,x)\leqslant \frac{p}{1-p}(d(gx_n,x_n)+d(fx_n,x_n)+d(x_n,gx_n))\\ +\frac{1}{1-p}d(x_n,fx_n). \quad (25)$$

Taking limit as $n\to\infty$ in (25), we obtain $\lim_{n\to\infty}d(x_n,x)=0$.

Therefore, the proof is completed, i.e, the common fixed point problem of f and g is well posed. □

4. Consequences

In this section, we give some consequences of our main results. For this purpose, we first recollect the basic concepts on G-metric spaces.

Definition 4.1 (See [14]). Let X be a non-empty set, $G:X\times X\times X\to\mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x,y,z)=0$ if $x=y=z$,

(G2) $0<G(x,x,y)$ for all $x,y\in X$ with $x\neq y$,

(G3) $G(x,x,y)\leqslant G(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$,

(G4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\cdots$(symmetry in all three variables),

(G5) $G(x,y,z)\leqslant G(x,a,a)+G(a,y,z)$ (rectangle inequality) for all $x,y,z,a\in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 4.2 (See [14]). A G-metric space (X,G) is said to be symmetric if $G(x,y,y)=G(y,x,x)$ for all $x,y\in X$.

For a better understanding of the subject we give the following examples of G-metrics:

Example 4.1 (See [14]). Let (X,d) be a metric space. The function $G:X\times X\times X\to[0,+\infty)$, defined by

$$G(x,y,z)=\max\{d(x,y),d(y,z),d(z,x)\},$$

for all $x,y,z\in X$, is a G-metric on X.

Example 4.2 (See [14]). Let $X=[0,\infty)$. The function $G:X\times X\times X\to[0,+\infty)$, defined by

$$G(x,y,z)=|x-y|+|y-z|+|z-x|,$$

for all $x,y,z\in X$, is a G-metric on X.

In their initial paper, Mustafa and Sims [14] also defined the basic topological concepts in G-metric spaces as follows:

Definition 4.3 (See [14]). Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x\in X$ if

$$\lim_{n,m\to+\infty}G(x,x_n,x_m)=0,$$

that is, for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $G(x,x_n,x_m)<\varepsilon$, for all $n,m\geqslant N$. We call x the limit of the sequence and write $x_n\to x$ or $\lim_{n\to+\infty}x_n=x$.

Proposition 4.1 (See [14]). *Let (X,G) be a G-metric space. The following are equivalent:*

(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n,x_n,x)\to 0$ as $n\to+\infty$,
(3) $G(x_n,x,x)\to 0$ as $n\to+\infty$,
(4) $G(x_n,x_m,x)\to 0$ as $n,m\to+\infty$.

Definition 4.4 (See [14]). Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $G(x_n,x_m,x_l)<\varepsilon$ for all $m,n,l\geqslant N$, that is, $G(x_n,x_m,x_l)\to 0$ as $n,m,l\to+\infty$.

Proposition 4.2 (See [14]). *Let (X,G) be a G-metric space. Then the followings are equivalent:*

(1) *the sequence $\{x_n\}$ is G-Cauchy,*
(2) *for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $G(x_n,x_m,x_m)<\varepsilon$, for all $m,n\geqslant N$.*

Definition 4.5 (See [14]). A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Notice that any G-metric space (X,G) induces a metric d_G on X defined by

$$d_G(x,y)=G(x,y,y)+G(y,x,x),\ \text{for all } x,y\in X. \quad (26)$$

Furthermore, (X,G) is G-complete if and only if (X,d_G) is complete.

Recently, Jleli and Samet [15] gave the following theorems.

Theorem 4.1 (See [15]). *Let (X,G) be a G-metric space. Let $d:X\times X\to[0,\infty)$ be the function defined by $d(x,y)=G(x,y,y)$. Then*

(1) *(X,d) is a quasi-metric space;*
(2) *$\{x_n\}\subset X$ is G-convergent to $x\in X$ if and only if $\{x_n\}$ is convergent to x in (X,d);*
(3) *$\{x_n\}\subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X,d);*

(4) (X,G) *is G-complete if and only if* (X,d) *is complete.*

Every quasi-metric induces a metric, that is, if (X,d) is a quasi-metric space, then the function $\delta : X \times X \to [0,\infty)$ defined by

$$\delta(x,y) = \max\{d(x,y), d(y,x)\} \tag{27}$$

is a metric on X [15].

Theorem 4.2 (See [15]). *Let* (X,G) *be a G-metric space. Let* $\delta : X \times X \to [0,\infty)$ *be the function defined by* $\delta(x,y) = \max\{G(x,y,y), G(y,x,x)\}$. *Then*

(1) (X,δ) *is a metric space;*
(2) $\{x_n\} \subset X$ *is G-convergent to* $x \in X$ *if and only if* $\{x_n\}$ *is convergent to* x *in* (X,δ)*;*
(3) $\{x_n\} \subset X$ *is G-Cauchy if and only if* $\{x_n\}$ *is Cauchy in* (X,δ)*;*
(4) (X,G) *is G-complete if and only if* (X,δ) *is complete.*

Now, we can give the following two corollaries on G-metric spaces. The first one is analogous to Theorem 4.4 of Popa and Patriciu [8].

Corollary 4.1. *Let* (X,G) *be a G-metric space and* $f,g : (X,G) \to (X,G)$ *satisfying*

$$F(G(fx,fy,fy), G(gx,gy,gy), G(gx,fx,fx), G(gy,fy,fy), G(gx,fy,fy), G(gy,fx,fx)) \leqslant 0, \tag{28}$$

for all $x,y \in X$, *where* $F \in \Gamma$. *If* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a G-complete metric subspace of* (X,G), *then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Consider the quasi-metric $d(x,y) = G(x,y,y)$ for all $x,y \in X$. We rewrite (28) as

$$F(d(fx,fy), d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)) \leqslant 0. \tag{29}$$

By Theorem 4.1, we also have that the quasi-metric space $(g(X),d)$ is complete. Then the result follows from Theorem 2.1. □

The notion of posedness of a common fixed point problem on G-metric spaces was introduced by Popa and Patriciu [8] as follows

Definition 4.6. Let (X,G) be a G-metric space and $f,g : (X,G) \to (X,G)$. The common fixed point problem of f and g is said to be well posed if

(1) f and g have a unique common fixed point,
(2) for any sequence $\{x_n\}$ in X with

$$\lim_{n\to\infty} G(x_n, fx_n, fx_n) = 0 \tag{30}$$

and

$$\lim_{n\to\infty} G(x_n, gx_n, gx_n) = 0, \tag{31}$$

then

$$\lim_{n\to\infty} G(x, x_n, x_n) = 0. \tag{32}$$

The following result is analogous to Theorem 5.5 of Popa and Patriciu [8].

Corollary 4.2. *Let* (X,G) *be a G-metric space. Suppose that the mappings* $f,g : (X,G) \to (X,G)$ *satisfy the hypotheses of Corollary* 4.1. *Assume also that F has the property* (F_p). *Then the common fixed point problem of f and g is well posed.*

Proof. Similarly, by considering the quasi-metric $d(x,y) = G(x,y,y)$ for all $x,y \in X$, the result follows easily from Theorems 3.1 and 4.1. □

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